TD 6 - Fonction [illegible]

(6.3.7 suite) Ainsi $J_n = 2\pi \frac{(2n-1)!}{(n-1)!^2}(2k)^{-2n+1}$

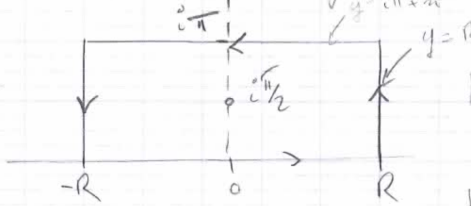
On conclut que : $\lim_{R\to\infty} J_n = \int_{-\infty}^{\infty} \frac{dx}{(k^2+x^2)^n} = 2\pi \frac{(2n-1)!}{(n-1)!^2}(2k)^{-2n+1}$

Ex 6.4

6.4.1 On veut calculer $I_a = \int_{-\infty}^{\infty} \frac{\exp(ax)}{ch(x)}\,dx$, $a\in\mathbb{R}$ $\xrightarrow{\text{intégrable}}$ $\underline{-1<a<1}$

Pôles $= Z = \{ i\frac{\pi}{2} + k\pi i \}$

On considère l'intégrale : $J_a = \int_{\gamma_R} \frac{\exp(az)}{ch(z)}\,dz$ $\quad R>1$



$H_R^1 = \int_{-R}^{R} \frac{\exp(ax)}{ch(x)}\,dx$

$H_R^2 = \int_0^{\pi} \frac{\exp(iax+aR)}{ch(ix+R)}\,i\,dx$

$H_R^3 = \int_R^{-R} \frac{\exp(ax+ai\pi)}{ch(i\pi+x)}\,dx$; $H_R^4 = \int_{\pi}^{0} \frac{\exp(iax+aR)}{ch(ix-R)}\,i\,dx$

On a : $|H_R^2| \leq \int_0^{\pi} \frac{\exp(aR)}{e^R - e^{-R}}\,dx = \frac{\exp(aR)}{e^R-e^{-R}}\pi \xrightarrow[R\to\infty]{} 0$

$|H_R^4| \leq \int_0^{\pi} \frac{\exp(-aR)}{e^R-e^{-R}}\,dx = \frac{\exp(-aR)}{e^R-e^{-R}}\pi \xrightarrow[R\to\infty]{} 0$

De plus, on remarque que : $ch(i\pi+x) = -ch(x)$ $\quad = \frac{1}{2}(e^{i\pi}e^{x} + e^{-i\pi}e^{-x}) = -\frac{1}{2}(e^x + e^{-x})$

$H_R^3 = -\int_R^{-R} \frac{\exp(ax+ai\pi)}{ch(x)}\,dx = \exp(ai\pi)\int_{-R}^{R} \frac{\exp(ax)}{ch(x)}\,dx$

On applique le théorème des résidus :

$J_a = 2\pi i \,\mathrm{Res}\left(\frac{\exp(az)}{ch(z)}, i\frac{\pi}{2}\right)$ Taylor-Lagrange

On rappelle que : $ch(z) = \underbrace{ch(i\tfrac{\pi}{2})}_{0} + (z-i\tfrac{\pi}{2})\,\underbrace{sh(i\tfrac{\pi}{2})}_{1} + (z-i\tfrac{\pi}{2})^2 ch(\xi)$ $\quad \exists \xi \in [z, i\tfrac{\pi}{2}]$

D'où : $\lim_{z\to i\frac{\pi}{2}} (z-i\tfrac{\pi}{2})\frac{\exp(az)}{ch(z)} = \lim_{z\to i\frac{\pi}{2}} \frac{\exp(az)(z-i\frac{\pi}{2})}{(z-i\frac{\pi}{2}) + (z-i\frac{\pi}{2})^2 ch(\xi)} = \exp(a\tfrac{i\pi}{2})$